Chapter 11

Inferences About Population Variances

Learning Objectives

1. Understand the importance of variance in a decision–making situation.

2 Understand the role of statistical inference in developing conclusions about the variance of a single population.

3. Know that the sampling distribution of $(n-1)\,s^2/\sigma^2$ has a chi–square distribution and be able to use this result to develop a confidence interval estimate of σ^2.

4. Be able to compute *p*–values using the chi–square distribution.

5. Know how to test hypotheses involving σ^2.

6. Understand the role of statistical inference in developing conclusions about two population variances.

7. Know that the sampling distribution of s_1^2/s_2^2 has an *F* distribution and be able to use this result to test hypotheses involving two population variances.

8. Be able to compute *p*–values using the *F* distribution.

Solutions

a. Management should be extremely pleased with the survey results: 40% + 46% = 86% of the ratings are very good to excellent, and 94% of the ratings are good or better. This does not look to be a Delta flight where significant changes are needed to improve the overall customer satisfaction ratings.

b. Some ways that could be used to collect data follow

1. a. 11.070

b. 27.488

c. 9.591

d. 23.209

e. 9.390

2. $s^2 = 25$

a. With 19 degrees of freedom $\chi^2_{.05} = 30.144$ and $\chi^2_{.95} = 10.117$:

$$\frac{19(25)}{30.144} \leq \sigma^2 \leq \frac{19(25)}{10.117}$$

$$15.76 \leq \sigma^2 \leq 46.95$$

b. With 19 degrees of freedom $\chi^2_{.025} = 32.852$ and $\chi^2_{.975} = 8.907$:

$$\frac{19(25)}{32.852} \leq \sigma^2 \leq \frac{19(25)}{8.907}$$

$$14.46 \leq \sigma^2 \leq 53.33$$

c. $3.8 \leq \sigma \leq 7.3$

3. $$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(16-1)(9.5)^2}{50} = 27.08$$

Degrees of freedom = (16 – 1) = 15.

Using χ^2 table, p–value is between .025 and .05.

Exact p–value using χ^2=27.08 is .0281.

p–value $\leq$ .05; reject H_0.

Critical value approach

$\chi^2_{.05} = 24.996$.

Reject H_0 if $\chi^2 \geq 24.996$.

27.08 > 24.996; reject H_0.

4. a. $n = 24$

$s^2 = .81$

With 24 – 1 = 23 degrees of freedom, $\chi^2_{.05}$ = 35.172 and $\chi^2_{.95}$ = 13.091.

$$\frac{23(.81)}{35.172} \leq \sigma^2 \leq \frac{23(.81)}{13.091}$$

$.53 \leq \sigma^2 \leq 1.42$

b. $.73 \leq \sigma \leq 1.19$

5. a. $\bar{x} = \frac{\sum_i x_i}{n} = \frac{13.6}{10} = 1.36$ or \$1,360,000

b. $s^2 = \frac{\sum_i (x_i - \bar{x})^2}{n-1} = \frac{8.36}{9} = .9293$

$s = \sqrt{0.9293} = .964$ or \$964,000

c. With df = n – 1 = 10 – 1 = 9 $\chi^2_{.025}$ = 19.023 $\chi^2_{.975}$ = 2.700

$$\frac{(10-1)(.9293)}{19.023} \leq \sigma^2 \leq \frac{(10-1)(.9293)}{2.700}$$

$.44 \leq \sigma^2 \leq 3.10$

d. $.66 \leq \sigma^2 \leq 1.76$

6. a. $\bar{x} = \frac{\sum x_i}{n} = \frac{25.6}{8} = 3.2$

The sample mean quarterly total return for General Electric is 3.2%. This is the estimate of the population mean percent total return per quarter for General Electric.

b. $s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = \frac{1773.6}{8-1} = 253.37$

$s = \sqrt{253.37} = 15.92$

c. With $df = (n - 1) = 7$ $\chi^2_{.025} = 16.013$ $\chi^2_{.975} = 1.690$

$$\frac{(n-1)s^2}{\chi_{.025}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi_{.975}}$$

$$\frac{(8-1)(253.37)}{16.013} \le \sigma^2 \le \frac{(8-1)(253.37)}{1.690}$$

$$110.76 \le \sigma^2 \le 1{,}049.47$$

d. $10.52 \le \sigma \le 32.40$

7. a. $\bar{x} = \frac{\sum x_i}{n} = \frac{656}{16} = 41$

The sample mean amount spent on a Halloween costume was $41.

b. $s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = \frac{8298}{16-1} = 553.2$

$s = \sqrt{553.2} = 23.52$

c. With $df = (n - 1) = 15$ $\chi^2_{.025} = 27.488$ $\chi^2_{.975} = 6.262$

$$\frac{(n-1)s^2}{\chi^2_{.025}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{.975}}$$

$$\frac{(16-1)(553.2)}{27.488} \le \sigma^2 \le \frac{(16-1)(553.2)}{6.262}$$

$$301.88 \le \sigma^2 \le 1325.14$$

$$17.37 \le \sigma \le 36.40$$

8. a. $\bar{x} = .78$

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1} = \frac{5.2230}{12-1} = .4748$$

b. $s = \sqrt{.4748} = .6891$

c. 11 degrees of freedom

$\chi^2_{.025} = 21.920$ $\chi^2_{.975} = 3.816$

$$\frac{(12-1).4748}{21.920} \le \sigma^2 \le \frac{(12-1).4748}{3.816}$$

$.2383 \le \sigma^2 \le 1.3687$

$.4882 \le \sigma \le 1.1699$

9. H_0: $\sigma^2 \le .0004$

H_a: $\sigma^2 > .0004$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(30-1)(.0005)}{.0004} = 36.25$$

Degrees of freedom = $n - 1 = 29$.

Using χ^2 table, p–value is greater than .10.

Exact p–value using $\chi^2 = 36.25$ is .1664.

p–value > .05; do not reject H_0. The product specification does not appear to be violated.

10. a. $\bar{x} = \frac{\sum x_i}{n} = \frac{1260}{15} = 84$

b. $s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1} = \frac{1662}{15-1} = 118.71$

c. $s = \sqrt{s^2} = \sqrt{118.71} = 10.90$

d. Hypothesis for $\sigma = 12$ is for $\sigma^2 = (12)^2 = 144$

H_0: $\sigma^2 = 144$

H_a: $\sigma^2 \neq 144$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(15-1)(118.71)}{144} = 11.54$$

Degrees of freedom = $n - 1 = 14$.

Using χ^2 table, p–value is greater than 2(1 – .90) = .20.

Exact p–value corresponding to χ^2 = 11.54 is 2(1 – .6431) = .7138.

p–value > .05; do not reject H_0. The hypothesis that the population standard deviation is 12; cannot be rejected.

11. a. $\bar{x} = \frac{\sum x_i}{n} = \frac{29,043}{51} = 569.47$

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1} = \frac{832,654.71}{50} = 16,653.09$$

$$s = \sqrt{16,653.09} = 129.05$$

b. H_0: $\sigma^2 = 14,660$

H_a: $\sigma^2 \neq 14,660$

c. $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(51-1)(16,653.09)}{14,660} = 56.80$

Degrees of freedom = $n - 1 = 50$.

Using χ^2 table, area in upper tail is greater than .10.

Two-tail p-value is greater than .20.

Exact two-tailed p-value corresponding to χ^2 = 56.80 with df = 50 is

2*CHISQ.DIST.RT(56.80, 50) = .4732.

p-value > .05; do not reject H_0. We cannot conclude the variance in GMAT scores for undergraduate students majoring in economics differs from the general population of GMAT test takers.

12. a. $s^2 = \dfrac{\Sigma(x_i - \overline{x})^2}{n-1} = .8106$

b. H_0: $\sigma^2 = .94$

H_a: $\sigma^2 \neq .94$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(12-1)(.8106)}{.94} = 9.49$$

Degrees of freedom = n – 1 = 11.

Using χ^2 table, area in tail is greater than .10.

Two–tail p–value is greater than .20.

Exact p–value corresponding to χ^2 = 9.49 is .8465.

p–value > .05, cannot reject H_0.

13. a. $F_{.05} = 3.33$

b. $F_{.025} = 2.76$

c. $F_{.01} = 4.50$

d. $F_{.10} = 1.94$

14. a. $F = \dfrac{s_1^2}{s_2^2} = \dfrac{5.8}{2.4} = 2.4$

Degrees of freedom 15 and 20.

Using F table, p–value is between .025 and .05.

Exact p–value corresponding to $F = 2.4$ is .0345.

p–value $\leq$ .05; reject H_0. Conclude $\sigma_1^2 > \sigma_2^2$.

b. $F_{.05} = 2.20$

Reject H_0 if $F \geq 2.20$

$2.4 \geq 2.20$; reject H_0. Conclude $\sigma_1^2 > \sigma_2^2$

15. a. Larger sample variance is s_1^2

$$F = \frac{s_1^2}{s_2^2} = \frac{8.2}{4} = 2.05$$

Degrees of freedom 20 and 25.

Using F table, area in tail is between .025 and .05.

Two–tail p–value is between .05 and .10 .

Exact p–value corresponding to $F = 2.05$ is .0904.

p–value > .05; do not reject H_0.

b. Because we have a two–tailed test:

$$F_{\alpha/2} = F_{.025} = 2.30$$

Reject H_0 if $F \geq 2.30$.

$2.05 < 2.30$; do not reject H_0 .

16. For this type of hypothesis test, we place the larger variance in the numerator. So the Fidelity variance is given the subscript of 1.

$$H_0: \sigma_1^2 \leq \sigma_2^2$$

$H_a: \sigma_1^2 > \sigma_2^2$

$$F = \frac{s_1^2}{s_2^2} = \frac{18.9^2}{15.0^2} = 1.59$$

Degrees of freedom in the numerator and denominator are both 59.

Using the *F* table, *p*–value is between .05 and .025.

Exact *p*–value corresponding to $F = 1.59$ is .0387.

p–value $\leq$ .05; reject H_0. We conclude that the Fidelity fund has a greater variance than the American Century fund.

17. a. Population 1 is four-year-old automobiles.

$H_0: \sigma_1^2 \leq \sigma_2^2$

$H_a: \sigma_1^2 > \sigma_2^2$

b. $$F = \frac{s_1^2}{s_2^2} = \frac{170^2}{100^2} = 2.89$$

Degrees of freedom 25 and 24.

Using *F* table, *p*–value is less than .01.

Exact *p*–value corresponding to $F = 2.89$ is .0057.

p–value $\leq$ .01; reject H_0. Conclude that 4 year old automobiles have a larger variance in annual repair costs compared to two-year-old automobiles. This is expected because older automobiles are more likely to have some more expensive repairs that lead to greater variance in annual repair costs.

18. We place the larger sample variance in the numerator. So, the Merrill Lynch variance is given the subscript of 1.

$H_0: \sigma_1^2 = \sigma_2^2$

$H_a: \sigma_1^2 \neq \sigma_2^2$

$$F = \frac{s_1^2}{s_2^2} = \frac{587^2}{489^2} = 1.44$$

Degrees of freedom 15 and 9.

Using *F* table, area in tail is greater than .10.

Two–tail *p*–value is greater than .20 .

Exact *p*–value corresponding to *F* = 1.44 is .5906.

p–value > .10; do not reject H_0. We cannot conclude there is a statistically significant difference between the variances for the two companies.

19. $H_0: \sigma_1^2 = \sigma_2^2$

$H_a: \sigma_1^2 \neq \sigma_2^2$

$s_1^2 = .01007$

$s_2^2 = .0012$

$$F = \frac{s_1^2}{s_2^2} = \frac{.01007}{.00120} = 8.39$$

Degrees of freedom 24 and 21.

Using *F* table, area in tail is less than .01.

Two–tail *p*–value is less than .02.

Exact *p*–value $\approx$ 0.

p–value $\leq$ .05; reject H_0. The process variances are significantly different.

Machine 1 offers the best opportunity for process quality improvements.

Note that the sample means are similar with the mean bag weights of approximately 1.5 kilograms. However, the process variances are significantly different.

20. $H_0: \sigma_1^2 = \sigma_2^2$

$H_a: \sigma_1^2 \neq \sigma_2^2$

$$F = \frac{s_1^2}{s_2^2} = \frac{11.1}{2.1} = 5.29$$

Degrees of freedom 25 and 24.

Using *F* table, area in tail is less than .01.

Two–tail *p*–value is less than .02 .

Exact *p*–value $\approx$ 0.

p–value $\leq$ .05; reject H_0. The population variances are not equal for seniors and managers.

21. a. Consider the talk time use as population 1 and Internet use as population 2.

$H_0: \sigma_1^2 \leq \sigma_2^2$

$H_a: \sigma_1^2 > \sigma_2^2$

b. $\bar{x}_1 = \frac{\sum_i x_i}{n_1} = \frac{336}{12} = 28.0$ hours

$$s_1 = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n_1 - 1}} = \sqrt{\frac{596.36}{12-1}} = 7.36 \bar{x}_2 = \frac{\sum_i x_i}{n_2} = \frac{108.1}{11} = 9.83 \text{ hours}$$

$$s_2 = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n_2 - 1}} = \sqrt{\frac{227.40}{11 - 1}} = 4.77$$

c. $F = \frac{s_1^2}{s_2^2} = \frac{7.36^2}{4.77^2} = 2.38$

Degrees of freedom $n_1 - 1 = 12 - 1 = 11$ and $n_2 = 11 - 1 = 10$.

The p–value is the upper–tail area at $F = 2.38$.

From the F table, the p–value is between .05 and .10.

Exact p–value corresponding to $F = 2.38$ is .09.

p–value > .05; do not reject H_0. There is not a statistically significant difference in the population variance in battery life for talk time and the population variance in battery life for the Internet use.

22. a. Population 1—Wet pavement.

$H_0: \sigma_1^2 \le \sigma_2^2$

$H_a: \sigma_1^2 > \sigma_2^2$

$$F = \frac{s_1^2}{s_2^2} = \frac{9.76^2}{4.88^2} = 4.00$$

Degrees of freedom 15 and 15.

Using F table, p–value is less than .01.

Exact p–value corresponding to $F = 4.00$ is .0054.

p–value $\le$ .05; reject H_0. Conclude that there is greater variability in stopping distances on wet pavement.

b. Drive carefully on wet pavement because of the variability in stopping distances.

23. a. $s^2 = (30)^2 = 900$

b. $\chi^2_{.05} = 30.144$ and $\chi^2_{.95} = 10.117$ (19 degrees of freedom)

$$\frac{(19)(900)}{30.144} \le \sigma^2 \le \frac{(19)(900)}{10.117}$$

$$567 \le \sigma^2 \le 1{,}690$$

c. $23.8 \le \sigma \le 41.1$

24. With 12 degrees of freedom:

$$\chi^2_{.025} = 23.337 \quad \chi^2_{.975} = 4.404$$

$$\frac{(12)(14.95)^2}{23.337} \le \sigma^2 \le \frac{(12)(14.95)^2}{4.404}$$

$$114.9 \le \sigma^2 \le 609$$

$$10.72 \le \sigma \le 24.68$$

25. a. $\bar{x} = \frac{\Sigma x_i}{n} = 260.16$

b. $s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = 4996.8$

$$s = \sqrt{4996.79} = 70.69$$

c. $\chi^2_{.025} = 32.852 \quad \chi^2_{.975} = 8.907$ (19 degrees of freedom)

$$\frac{(20-1)(4996.8)}{32.852} \le \sigma^2 \le \frac{(20-1)(4996.8)}{8.907}$$

$$2{,}890 \le \sigma^2 \le 10{,}659$$

$$53.76 \le \sigma \le 103.24$$

26. a. H_0: $\sigma^2 \le .0625$

H_a: $\sigma^2 \le .0625$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(15-1)(.35)^2}{.0625} = 27.44$$

Degrees of freedom = $n - 1 = 14$.

Using χ^2 table, p– value is between .01 and .025.

Exact p–value corresponding to $\chi^2 = 27.44$ is .0169.

p–value $\leq$ .10; reject H_0. Variance exceeds maximum variance requirement.

b. $\chi^2_{.05} = 23.685$

$\chi^2_{.95} = 6.571$ (14 degrees of freedom)

$$\frac{(14)(.35)^2}{23.685} \leq \sigma^2 \leq \frac{(14)(.35)^2}{6.571}$$

$.07241 \leq \sigma^2 \leq .26100$

27. H_0: $\sigma^2 \leq 16$

H_a: $\sigma^2 > 16$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(41-1)(4.5)^2}{16} = 50.63$$

Degrees of freedom = $n - 1 = 40$.

Using χ^2 table, p– value is greater than .10.

Exact p–value corresponding to $\chi^2 = 50.63$ is .1210.

p–value > .05; do not reject H_0. The population variance does not appear to be exceeding the standard.

28. H_0: $\sigma^2 \leq 1$

H_a: $\sigma^2 > 1$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(22-1)(1.5)}{1} = 31.50$$

Degrees of freedom = n – 1 = 21

Using χ^2 table, p–value is between .05 and .10

Exact p–value corresponding to χ^2 = 31.50 is .0657

p–value $\leq$ .10; reject H_0. Conclude that $\sigma^2 > 1$.

29. $$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = \frac{101.56}{9-1} = 12.69$$

H_0: $\sigma^2 = 10$

H_a: $\sigma^2 \neq 10$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(9-1)(12.69)}{10} = 10.16$$

Degrees of freedom = n – 1 = 8.

Using χ^2 table, area in tail is greater than .10.

Two–tail p– value is greater than .20.

Exact p–value corresponding to χ^2 = 10.16 is .5080.

p–value > .10; do not reject H_0.

30. a. Try n = 15

$\chi^2_{.025}$ = 26.119

$\chi^2_{.975}$ = 5.629 (14 degrees of freedom)

$$\frac{(14)(64)}{26.119} \le \sigma^2 \le \frac{(14)(64)}{5.629}$$

$$34.3 \le \sigma^2 \le 159.2$$

$$5.86 \le \sigma \le 12.62$$

$\therefore$ A sample size of 15 was used.

b. $n = 25$; expect the width of the interval to be smaller.

$$\chi^2_{.05} = 39.364$$

$$\chi^2_{.975} = 12.401 \text{ (24 degrees of freedom)}$$

$$\frac{(24)(8)^2}{39.364} \le \sigma^2 \le \frac{(24)(8)^2}{12.401}$$

$$39.02 \le \sigma^2 \le 123.86$$

$$6.25 \le \sigma \le 11.13$$

31. $H_0\colon \sigma_1^2 = \sigma_2^2$

$H_a\colon \sigma_1^2 \ne \sigma_2^2$

Population 1 is women's scores.

$$F = \frac{s_1^2}{s_2^2} = \frac{2.4623^2}{2.2118^2} = 1.24$$

Degrees of freedom 19 and 29

Using Excel of Minitab, we find the exact two–tail p–value corresponding to $F = 1.24$ is .5876.

p–value > .10; do not reject H_0. There is not a statistically significant difference in the variances. We cannot conclude that there is a difference in the variability of golf scores for male and female professional golfers.

32. H_0: $\sigma_1^2 = \sigma_2^2$

H_a: $\sigma_1^2 \neq \sigma_2^2$

Use critical value approach because F tables do not have 351 and 72 degrees of freedom.

$F_{.025} = 1.47$

Reject H_0 if $F \geq 1.47$.

$$F = \frac{s_1^2}{s_2^2} = \frac{.940^2}{.797^2} = 1.39$$

$F < 1.47$; do not reject H_0. We are not able to conclude students who complete the course and students who drop out have different variances of grade point averages.

33. H_0: $\sigma_1^2 = \sigma_2^2$

H_a: $\sigma_1^2 \neq \sigma_2^2$

Population 1 has the larger sample variance.

$$F = \frac{s_1^2}{s_2^2} = \frac{5.4}{2.3} = 2.35$$

Degrees of freedom 15 and 15.

Using F table, area in tail is between .05 and .10.

Two–tail p–value is between .10 and .20.

Exact p–value corresponding to $F = 2.35$ is .1087.

p–value > .10; do not reject H_0. Cannot conclude that there is a difference between the population variances.

34. $H_0\colon \sigma_1^2 \le \sigma_2^2$

$H_1\colon \sigma_1^2 > \sigma_2^2$

$$F = \frac{s_1^2}{s_2^2} = \frac{25}{12} = 2.08$$

Degrees of freedom 30 and 24

Using the F table, the area in the tail (which corresponds to the p-value) is between .025 and .05.

Exact p–value corresponding to $F = 2.08$ is .0348.

p–value $\le$ .10; reject H_0. Conclude that the population variances has decreased because of the lean process improvement.