Calculus(II) HW14 (06/11)

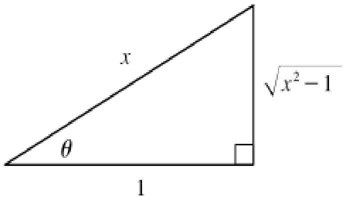
Sec.7.3 # 5

Evaluate the integral.

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx$$

[Solution]

Let $x = \sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ for the relevant values of θ, so



$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\sec\theta\,\tan\theta\,d\theta = \int \tan^2\theta\,\cos^3\theta\,d\theta$$

$$= \int \sin^2\theta\,\cos\theta\,d\theta \overset{s}{=} \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3\theta + C$$

$$= \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = \frac{1}{3}\frac{(x^2-1)^{3/2}}{x^3} + C$$

Sec.7.3 # 15

Evaluate the integral.

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

[Solution]

Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, $x = 0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^2\sin^2\theta\,(a\cos\theta)\,a\cos\theta\,d\theta = a^4\int_0^{\pi/2}\sin^2\theta\,\cos^2\theta\,d\theta$$

$$= a^4\int_0^{\pi/2}\left[\tfrac{1}{2}(2\sin\theta\,\cos\theta)\right]^2 d\theta = \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{a^4}{8}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{a^4}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi}{16}a^4$$

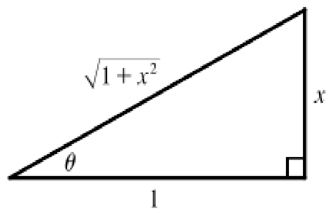
Sec.7.3 # 19

Evaluate the integral.

$$\int \frac{\sqrt{1+x^2}}{x}\,dx$$

[Solution]

Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\, d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so



$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta$$

$$= \int (\csc\theta + \sec\theta\,\tan\theta)\, d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.39]}$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

Sec.7.3 # 26

Evaluate the integral.

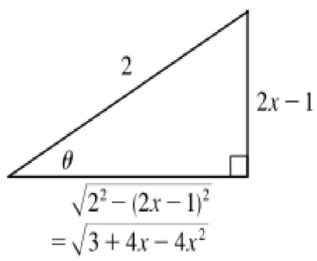
$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$$

[Solution]

$3+4x-4x^2 = -(4x^2-4x+1)+4 = 2^2-(2x-1)^2$.

Let $2x-1 = 2\sin\theta$, so $2\,dx = 2\cos\theta\, d\theta$ and $\sqrt{3+4x-4x^2} = 2\cos\theta$.

Then



$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx = \int \frac{\left[\frac{1}{2}(1+2\sin\theta)\right]^2}{(2\cos\theta)^3}\cos\theta\, d\theta$$

$$= \frac{1}{32}\int \frac{1+4\sin\theta+4\sin^2\theta}{\cos^2\theta}\, d\theta = \frac{1}{32}\int(\sec^2\theta + 4\tan\theta\,\sec\theta + 4\tan^2\theta)\, d\theta$$

$$= \tfrac{1}{32}\int[\sec^2\theta + 4\tan\theta\,\sec\theta + 4(\sec^2\theta - 1)]\, d\theta$$

$$= \tfrac{1}{32}\int(5\sec^2\theta + 4\tan\theta\,\sec\theta - 4)\, d\theta = \tfrac{1}{32}(5\tan\theta + 4\sec\theta - 4\theta) + C$$

$$= \frac{1}{32}\left[5\cdot\frac{2x-1}{\sqrt{3+4x-4x^2}} + 4\cdot\frac{2}{\sqrt{3+4x-4x^2}} - 4\cdot\sin^{-1}\left(\frac{2x-1}{2}\right)\right] + C$$

$$= \frac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C$$