Calculus(II) HW5 (3/26)

Sec.5.1 # 8

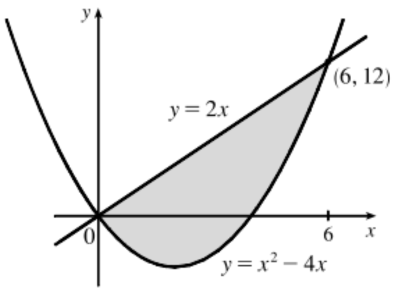
5–12 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

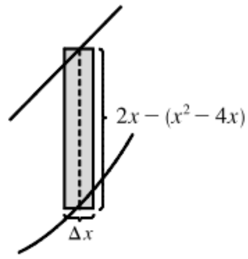
$y = x^2 - 4x, \quad y = 2x$

[Solution]

The curves intesect when $x^2 - 4x = 2x \quad \Rightarrow \quad x^2 - 6x = 0 \quad \Rightarrow \quad x(x-6) = 0 \quad \Rightarrow \quad x = 0$ or 6.

$$A = \int_0^6 [2x - (x^2 - 4x)]\, dx$$
$$= \int_0^6 (6x - x^2)\, dx = \left[3x^2 - \tfrac{1}{3}x^3\right]_0^6$$
$$= \left[3(6)^2 - \tfrac{1}{3}(6)^3\right] - (0 - 0)$$
$$= 108 - 72 = 36$$





Sec.5.1 # 12

5–12 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

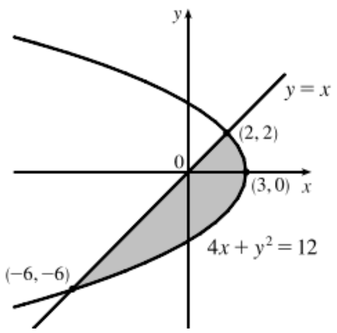
$4x + y^2 = 12, \quad x = y$

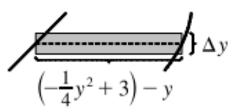
[Solution]

$4x + x^2 = 12 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow$

$x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy$$
$$= \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2}$$
$$= \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18)$$
$$= 22 - \tfrac{2}{3} = \tfrac{64}{3}$$





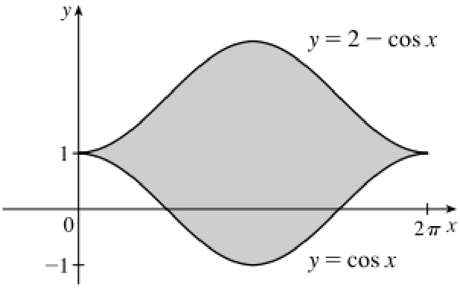
Sec.5.1 # 16

13–28 Sketch the region enclosed by the given curves and find its area.

$y = \cos x, \quad y = 2 - \cos x, \quad 0 \leqslant x \leqslant 2\pi$

[Solution]

$$A = \int_0^{2\pi} [(2 - \cos x) - \cos x]\, dx$$
$$= \int_0^{2\pi} (2 - 2\cos x)\, dx$$
$$= \Big[2x - 2\sin x\Big]_0^{2\pi}$$
$$= (4\pi - 0) - 0 = 4\pi$$



Sec.5.1 # 18

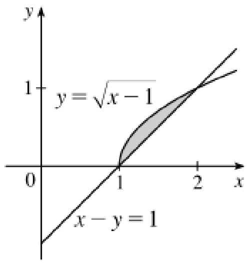
13–28 Sketch the region enclosed by the given curves and find its area.

$y = \sqrt{x-1}, \quad x - y = 1$

[Solution]

The curves intersect when $\sqrt{x-1} = x - 1 \;\Rightarrow$
$x - 1 = x^2 - 2x + 1 \;\Leftrightarrow\; 0 = x^2 - 3x + 2 \;\Leftrightarrow$
$0 = (x-1)(x-2) \;\Leftrightarrow\; x = 1$ or 2.

$$A = \int_1^2 \left[\sqrt{x-1} - (x-1)\right] dx$$
$$= \left[\tfrac{2}{3}(x-1)^{3/2} - \tfrac{1}{2}(x-1)^2\right]_1^2 = \left(\tfrac{2}{3} - \tfrac{1}{2}\right) - (0 - 0) = \tfrac{1}{6}$$



Sec.5.1 # 24

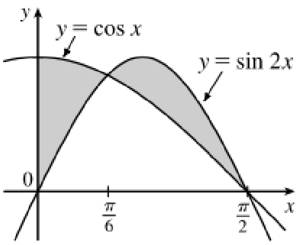
13–28 Sketch the region enclosed by the given curves and find its area.

$y = \cos x, \quad y = \sin 2x, \quad x = 0, \quad x = \pi/2$

[Solution]

Notice that $\cos x = \sin 2x = 2\sin x \cos x \;\Leftrightarrow\; 2\sin x\cos x - \cos x = 0 \;\Leftrightarrow\; \cos x\,(2\sin x - 1) = 0 \;\Leftrightarrow$
$2\sin x = 1$ or $\cos x = 0 \;\Leftrightarrow\; x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx$$
$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$



Sec.5.1 # 28

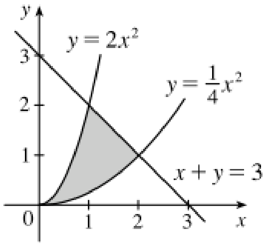
13–28 Sketch the region enclosed by the given curves and find its area.

$y = \frac{1}{4}x^2, \quad y = 2x^2, \quad x + y = 3, \quad x \geqslant 0$

[Solution]

$\frac{1}{4}x^2 = -x+3 \quad \Leftrightarrow \quad x^2 + 4x - 12 = 0 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow \quad x = -6$ or 2 and $2x^2 = -x + 3 \quad \Leftrightarrow$
$2x^2 + x - 3 = 0 \quad \Leftrightarrow \quad (2x+3)(x-1) = 0 \quad \Leftrightarrow \quad x = -\frac{3}{2}$ or 1, so for $x \geq 0$,

$$\begin{aligned} A &= \int_0^1 \left(2x^2 - \tfrac{1}{4}x^2\right) dx + \int_1^2 \left[(-x+3) - \tfrac{1}{4}x^2\right] dx \\ &= \int_0^1 \tfrac{7}{4}x^2\, dx + \int_1^2 \left(-\tfrac{1}{4}x^2 - x + 3\right) dx \\ &= \left[\tfrac{7}{12}x^3\right]_0^1 + \left[-\tfrac{1}{12}x^3 - \tfrac{1}{2}x^2 + 3x\right]_1^2 \\ &= \tfrac{7}{12} + \left(-\tfrac{2}{3} - 2 + 6\right) - \left(-\tfrac{1}{12} - \tfrac{1}{2} + 3\right) = \tfrac{3}{2} \end{aligned}$$



Sec.5.2 # 3

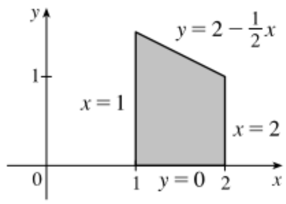
1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

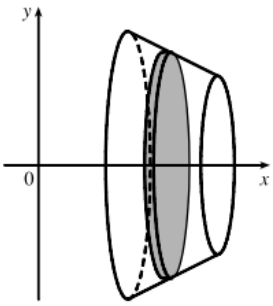
$y = 2 - \frac{1}{2}x, \ y = 0, \ x = 1, \ x = 2;$ about the x-axis

[Solution]

A cross-section is a disk with radius $2 - \frac{1}{2}x$, so its area is $A(x) = \pi\left(2 - \frac{1}{2}x\right)^2$.

$$\begin{aligned} V &= \int_1^2 A(x)\, dx = \int_1^2 \pi\left(2 - \tfrac{1}{2}x\right)^2 dx \\ &= \pi \int_1^2 \left(4 - 2x + \tfrac{1}{4}x^2\right) dx \\ &= \pi\left[4x - x^2 + \tfrac{1}{12}x^3\right]_1^2 \\ &= \pi\left[\left(8 - 4 + \tfrac{8}{12}\right) - \left(4 - 1 + \tfrac{1}{12}\right)\right] \\ &= \pi\left(1 + \tfrac{7}{12}\right) = \tfrac{19}{12}\pi \end{aligned}$$





Sec.5.2 # 5

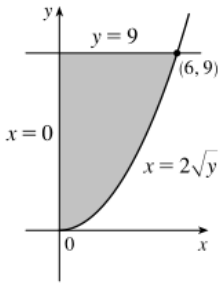
1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

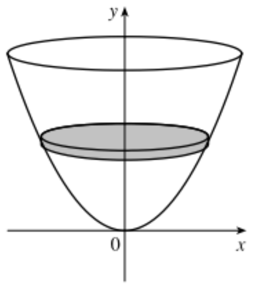
$x = 2\sqrt{y}, \ x = 0, \ y = 9;$ about the y-axis

[Solution]

A cross-section is a disk with radius $2\sqrt{y}$, so its area is $A(y) = \pi\left(2\sqrt{y}\right)^2$.

$$V = \int_0^9 A(y)\,dy = \int_0^9 \pi\left(2\sqrt{y}\right)^2 dy = 4\pi \int_0^9 y\,dy$$
$$= 4\pi\left[\tfrac{1}{2}y^2\right]_0^9 = 2\pi(81) = 162\pi$$





Sec.5.2 # 14

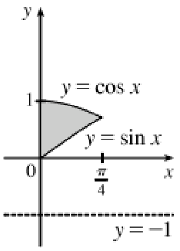
1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

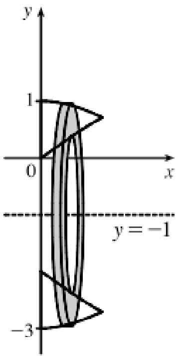
$y = \sin x,\ y = \cos x,\ 0 \leqslant x \leqslant \pi/4;$ about $y = -1$

[Solution]

A cross-section is a washer with inner radius $\sin x - (-1)$ and outer radius $\cos x - (-1)$, so its area is

$$A(x) = \pi\left[(\cos x + 1)^2 - (\sin x + 1)^2\right]$$
$$= \pi(\cos^2 x + 2\cos x - \sin^2 x - 2\sin x)$$
$$= \pi(\cos 2x + 2\cos x - 2\sin x).$$
$$V = \int_0^{\pi/4} A(x)\,dx = \int_0^{\pi/4} \pi(\cos 2x + 2\cos x - 2\sin x)\,dx$$
$$= \pi\left[\tfrac{1}{2}\sin 2x + 2\sin x + 2\cos x\right]_0^{\pi/4}$$
$$= \pi\left[\left(\tfrac{1}{2} + \sqrt{2} + \sqrt{2}\right) - (0 + 0 + 2)\right] = \left(2\sqrt{2} - \tfrac{3}{2}\right)\pi$$





Sec.5.2 # 16

1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

$y = x^2,\ x = y^2;$ about $x = -1$

[Solution]

$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$$V = \int_0^1 \pi \left\{ \left[\sqrt{y} - (-1) \right]^2 - \left[y^2 - (-1) \right]^2 \right\} dy = \pi \int_0^1 \left[\left(\sqrt{y} + 1 \right)^2 - (y^2 + 1)^2 \right] dy$$

$$= \pi \int_0^1 \left(y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1 \right) dy = \pi \int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2 \right) dy$$

$$= \pi \left[\tfrac{1}{2} y^2 + \tfrac{4}{3} y^{3/2} - \tfrac{1}{5} y^5 - \tfrac{2}{3} y^3 \right]_0^1 = \pi \left(\tfrac{1}{2} + \tfrac{4}{3} - \tfrac{1}{5} - \tfrac{2}{3} \right) = \tfrac{29}{30} \pi$$