Calculus(II) HW9 (05/07)

**Sec.6.2* # 62**

**61–64** Use logarithmic differentiation to find the derivative of the function.

$$y = \sqrt{x}\, e^{x^2}(x^2+1)^{10}$$

[Solution]

$$y = \sqrt{x}\, e^{x^2}\left(x^2+1\right)^{10} \quad\Rightarrow\quad \ln y = \ln\sqrt{x} + \ln e^{x^2} + \ln(x^2+1)^{10} \quad\Rightarrow\quad \ln y = \tfrac{1}{2}\ln x + x^2 + 10\ln(x^2+1) \quad\Rightarrow$$

$$\frac{1}{y}y' = \frac{1}{2}\cdot\frac{1}{x} + 2x + 10\cdot\frac{1}{x^2+1}\cdot 2x \quad\Rightarrow\quad y' = \sqrt{x}\, e^{x^2}(x^2+1)^{10}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$$

**Sec.6.2* # 63**

**61–64** Use logarithmic differentiation to find the derivative of the function.

$$y = \sqrt{\frac{x-1}{x^4+1}}$$

[Solution]

$$y = \sqrt{\frac{x-1}{x^4+1}} \quad\Rightarrow\quad \ln y = \ln\left(\frac{x-1}{x^4+1}\right)^{1/2} \quad\Rightarrow\quad \ln y = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x^4+1) \quad\Rightarrow$$

$$\frac{1}{y}y' = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x^4+1}\cdot 4x^3 \quad\Rightarrow\quad y' = y\left(\frac{1}{2(x-1)} - \frac{2x^3}{x^4+1}\right) \quad\Rightarrow\quad y' = \sqrt{\frac{x-1}{x^4+1}}\left(\frac{1}{2x-2} - \frac{2x^3}{x^4+1}\right)$$

**Sec.6.2* # 68**

Evaluate the integral.

$$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$$

[Solution]

$$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4)$$

$$= \tfrac{85}{2} + \ln\tfrac{9}{4}$$

**Sec.6.2* # 70**

> Evaluate the integral.
>
> $\int_e^6 \frac{dx}{x \ln x}$

[Solution]

. Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, so $\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{\ln 6} = \ln\ln 6 - \ln 1 = \ln\ln 6$

**Sec.6.2* # 73**

> Evaluate the integral.
>
> $\int \frac{\sin 2x}{1 + \cos^2 x}\,dx$

[Solution]

$\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x \cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$2I = -2\int \frac{u\,du}{1+u^2} = -2 \cdot \frac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C$.

*Or:* Let $u = 1 + \cos^2 x$.

**Sec.6.3* # 6**

> Solve each equation for $x$.
>
> (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

[Solution]

(a) $\ln(x^2-1) = 3 \;\Leftrightarrow\; x^2 - 1 = e^3 \;\Leftrightarrow\; x^2 = 1 + e^3 \;\Leftrightarrow\; x = \pm\sqrt{1+e^3}$.

(b) $e^{2x} - 3e^x + 2 = 0 \;\Leftrightarrow\; (e^x - 1)(e^x - 2) = 0 \;\Leftrightarrow\; e^x = 1$ or $e^x = 2 \;\Leftrightarrow\; x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

**Sec.6.3* # 10**

> Solve each equation for $x$.
>
> $10(1 + e^{-x})^{-1} = 3$

[Solution]

$10(1+e^{-x})^{-1} = 3 \;\Leftrightarrow\; (1+e^{-x})^{-1} = \frac{3}{10} \;\Leftrightarrow\; 1 + e^{-x} = \frac{10}{3} \;\Leftrightarrow\; e^{-x} = \frac{7}{3} \;\Leftrightarrow\; -x = \ln\frac{7}{3} \;\Rightarrow$

$x = -\ln\frac{7}{3} = \ln\left(\frac{7}{3}\right)^{-1} = \ln\frac{3}{7}$

**Sec.6.3* # 21**

Find (a) the domain of $f$ and (b) $f^{-1}$ and its domain.
$f(x) = \sqrt{3 - e^{2x}}$

[Solution]

(a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \;\Rightarrow\; e^{2x} \leq 3 \;\Rightarrow\; 2x \leq \ln 3 \;\Rightarrow\; x \leq \frac{1}{2}\ln 3$.

Thus, the domain of $f$ is $(-\infty, \frac{1}{2}\ln 3]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow\; y^2 = 3 - e^{2x} \;\Rightarrow\; e^{2x} = 3 - y^2 \;\Rightarrow\; 2x = \ln(3 - y^2) \;\Rightarrow$ $x = \frac{1}{2}\ln(3 - y^2)$. Interchange $x$ and $y$: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of $f^{-1}$, we must have $3 - x^2 > 0 \;\Rightarrow\; x^2 < 3 \;\Rightarrow\; |x| < \sqrt{3} \;\Rightarrow\; -\sqrt{3} < x < \sqrt{3} \;\Rightarrow\; 0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of $f^{-1}$, $[0, \sqrt{3}\,)$, equals the range of $f$.

**Sec.6.3* # 26**

Find the inverse function.

$$y = \frac{1 - e^{-x}}{1 + e^{-x}}$$

[Solution]

$y = f(x) = \dfrac{1 - e^{-x}}{1 + e^{-x}} \;\Rightarrow\; y(1 + e^{-x}) = 1 - e^{-x} \;\Rightarrow\; y + ye^{-x} = 1 - e^{-x} \;\Rightarrow\; ye^x + y = e^x - 1$ [multiply each term by $e^x$] $\Rightarrow\; ye^x - e^x = -y - 1 \;\Rightarrow\; e^x(y - 1) = -y - 1 \;\Rightarrow\; e^x = \dfrac{1 + y}{1 - y} \;\Rightarrow\; x = \ln\left(\dfrac{1 + y}{1 - y}\right)$.

Interchange $x$ and $y$: $y = \ln\left(\dfrac{1 + x}{1 - x}\right)$. So $f^{-1}(x) = \ln\left(\dfrac{1 + x}{1 - x}\right)$.

**Sec.6.3* # 29**

Find the limit.

$$\lim_{x \to 2^+} e^{3/(2-x)}$$

[Solution]

Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\displaystyle\lim_{x \to 2+} e^{3/(2-x)} = \lim_{t \to -\infty} e^t = 0$ by (6).