Calculus(II) HW13 (06/04)

Sec.7.1 # 23

Evaluate the integral.

$$\int_0^{1/2} x \cos \pi x \, dx$$

[Solution]

Let $u = x$, $dv = \cos \pi x \, dx \quad \Rightarrow \quad du = dx$, $v = \frac{1}{\pi} \sin \pi x$. By (6),

$$\int_0^{1/2} x \cos \pi x \, dx = \left[\frac{1}{\pi} x \sin \pi x \right]_0^{1/2} - \int_0^{1/2} \frac{1}{\pi} \sin \pi x \, dx = \frac{1}{2\pi} - 0 - \frac{1}{\pi} \left[-\frac{1}{\pi} \cos \pi x \right]_0^{1/2}$$

$$= \frac{1}{2\pi} + \frac{1}{\pi^2}(0 - 1) = \frac{1}{2\pi} - \frac{1}{\pi^2} \text{ or } \frac{\pi - 2}{2\pi^2}$$

Sec.7.1 # 30

Evaluate the integral.

$$\int_1^{\sqrt{3}} \arctan(1/x) \, dx$$

[Solution]

Let $u = \arctan(1/x)$, $dv = dx \quad \Rightarrow \quad du = \frac{1}{1 + (1/x)^2} \cdot \frac{-1}{x^2} \, dx = \frac{-dx}{x^2 + 1}$, $v = x$. By (6),

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = \left[x \arctan\left(\frac{1}{x}\right) \right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x \, dx}{x^2 + 1} = \sqrt{3} \frac{\pi}{6} - 1 \cdot \frac{\pi}{4} + \frac{1}{2} \left[\ln(x^2 + 1) \right]_1^{\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2} \ln \frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2} \ln 2$$

Sec.7.2 # 4

Evaluate the integral.

$$\int_0^{\pi/2} \sin^5 x \, dx$$

[Solution]

$$\int_0^{\pi/2} \sin^5 x \, dx = \int_0^{\pi/2} \sin^4 x \, \sin x \, dx = \int_0^{\pi/2} (1 - \cos^2 x)^2 \, \sin x \, dx \stackrel{c}{=} \int_1^0 (1 - u^2)^2 (-du)$$

$$= \int_0^1 (1 - 2u^2 + u^4) \, du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5} \right) - 0 = \frac{15 - 10 + 3}{15} = \frac{8}{15}$$

Sec.7.2 # 8

Evaluate the integral.

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx$$

[Solution]

Let $y = \sqrt{x}$, so that $dy = \dfrac{1}{2\sqrt{x}}\,dx$ and $dx = 2y\,dy$. Then

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx = \int \frac{\sin^3 y}{y}(2y\,dy) = 2\int \sin^3 y\,dy = 2\int \sin^2 y\,\sin y\,dy = 2\int (1-\cos^2 y)\,\sin y\,dy$$

$$\overset{c}{=} 2\int (1-u^2)(-du) = 2\int (u^2-1)\,du = 2\left(\tfrac{1}{3}u^3 - u\right) + C = \tfrac{2}{3}\cos^3 y - 2\cos y + C$$

$$= \tfrac{2}{3}\cos^3(\sqrt{x}) - 2\cos\sqrt{x} + C$$

Sec.7.2 # 11

Evaluate the integral.

$$\int_0^{\pi/2} \sin^2 x\,\cos^2 x\,dx$$

[Solution]

$$\int_0^{\pi/2} \sin^2 x\,\cos^2 x\,dx = \int_0^{\pi/2} \tfrac{1}{4}(4\sin^2 x\,\cos^2 x)\,dx = \int_0^{\pi/2} \tfrac{1}{4}(2\sin x\,\cos x)^2 dx = \tfrac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx$$

$$= \tfrac{1}{4}\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 4x)\,dx = \tfrac{1}{8}\int_0^{\pi/2}(1-\cos 4x)\,dx = \tfrac{1}{8}\left[x - \tfrac{1}{4}\sin 4x\right]_0^{\pi/2} = \tfrac{1}{8}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{16}$$

Sec.7.2 # 12

Evaluate the integral.

$$\int_0^{\pi/2} (2 - \sin\theta)^2\,d\theta$$

[Solution]

$$\int_0^{\pi/2}(2-\sin\theta)^2\,d\theta = \int_0^{\pi/2}(4 - 4\sin\theta + \sin^2\theta)\,d\theta = \int_0^{\pi/2}\left[4 - 4\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= \int_0^{\pi/2}\left(\tfrac{9}{2} - 4\sin\theta - \tfrac{1}{2}\cos 2\theta\right)d\theta = \left[\tfrac{9}{2}\theta + 4\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= \left(\tfrac{9\pi}{4} + 0 - 0\right) - (0 + 4 - 0) = \tfrac{9}{4}\pi - 4$$

Sec.7.2 # 21

Evaluate the integral.

$\int \tan x \sec^3 x\, dx$

[Solution]

$$\int \tan x \sec^3 x\, dx = \int \tan x \sec x \sec^2 x\, dx = \int u^2\, du \qquad [u = \sec x,\, du = \sec x \tan x\, dx]$$
$$= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sec^3 x + C$$

Sec.7.2 # 26

Evaluate the integral.

$\int_0^{\pi/4} \sec^6\theta \tan^6\theta\, d\theta$

[Solution]

$$\int_0^{\pi/4} \sec^6\theta \tan^6\theta\, d\theta = \int_0^{\pi/4} \tan^6\theta \sec^4\theta \sec^2\theta\, d\theta = \int_0^{\pi/4} \tan^6\theta(1+\tan^2\theta)^2 \sec^2\theta\, d\theta$$
$$= \int_0^1 u^6(1+u^2)^2\, du \qquad \begin{bmatrix} u = \tan\theta, \\ du = \sec^2\theta\, d\theta \end{bmatrix}$$
$$= \int_0^1 u^6(u^4+2u^2+1)\, du = \int_0^1 (u^{10}+2u^8+u^6)\, du$$
$$= \left[\tfrac{1}{11}u^{11} + \tfrac{2}{9}u^9 + \tfrac{1}{7}u^7\right]_0^1 = \tfrac{1}{11} + \tfrac{2}{9} + \tfrac{1}{7} = \tfrac{63+154+99}{693} = \tfrac{316}{693}$$