Calculus(II) HW11 (05/21)

Sec.6.6 # 3

Find the exact value of each expression.

(a) $\csc^{-1}\sqrt{2}$ (b) $\cos^{-1}(\sqrt{3}/2)$

[Solution]

(a) $\csc^{-1}\sqrt{2} = \frac{\pi}{4}$ since $\csc\frac{\pi}{4} = \sqrt{2}$.

(b) $\cos^{-1}\left(\sqrt{3}/2\right) = \frac{\pi}{6}$ because $\cos\frac{\pi}{6} = \sqrt{3}/2$ and $\frac{\pi}{6}$ is in $[0, \pi]$.

Sec.6.6 # 7

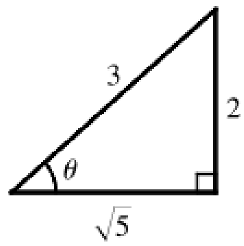
Find the exact value of each expression.

$\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

[Solution]

Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$ [see the figure].

Then $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \tan\theta = \dfrac{2}{\sqrt{5}}$.



Sec.6.6 # 21

Prove that $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2-1}}$.

[Solution]

Let $y = \csc^{-1}x$. Then $\csc y = x \;\Rightarrow\; -\csc y\,\cot y\,\dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\csc y\,\cot y} = -\dfrac{1}{\csc y\sqrt{\csc^2 y - 1}} = -\dfrac{1}{x\sqrt{x^2-1}}$. Note that $\cot y \geq 0$ on the domain of $\csc^{-1}x$.

Sec.6.6 # 25

Find the derivative of the function. Simplify where possible. $y = \sin^{-1}(2x + 1)$

[Solution]

$y = \sin^{-1}(2x+1) \quad \Rightarrow$

$$y' = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot \frac{d}{dx}(2x+1) = \frac{1}{\sqrt{1-(4x^2+4x+1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2-4x}} = \frac{1}{\sqrt{-x^2-x}}$$

Sec.6.6 # 30

Find the derivative of the function. Simplify where possible. $g(x) = \sqrt{x^2 - 1}\,\sec^{-1}x$

[Solution]

$$g(x) = \sqrt{x^2-1}\,\sec^{-1}x \quad \Rightarrow \quad g'(x) = \sqrt{x^2-1} \cdot \frac{1}{x\sqrt{x^2-1}} + \sec^{-1}x \cdot \frac{1}{2}\left(x^2-1\right)^{-1/2}(2x) = \frac{1}{x} + \frac{x\sec^{-1}x}{\sqrt{x^2-1}}$$

$$\left[\text{or } \frac{\sqrt{x^2-1} + x^2\sec^{-1}x}{x\sqrt{x^2-1}}\right]$$

Sec.6.6 # 34

Find the derivative of the function. Simplify where possible. $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$

[Solution]

$$y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}} = \tan^{-1}\left(\frac{x}{a}\right) + \tfrac{1}{2}\ln(x-a) - \tfrac{1}{2}\ln(x+a) \quad \Rightarrow$$

$$y' = \frac{a}{x^2+a^2} + \frac{1/2}{x-a} - \frac{1/2}{x+a} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2} = \frac{2ax^2}{x^4-a^4}$$

Sec.6.6 # 61

Evaluate the integral.

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

[Solution]

Let $u = \sin^{-1} x$, so $du = \dfrac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

Sec.6.6 # 64

Evaluate the integral.

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$$

[Solution]

Let $u = -\cos x$. Then $du = \sin x\,dx$, so

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \int_{-1}^{0} \frac{1}{1+u^2}\,du = \left[\tan^{-1} u\right]_{-1}^{0} = \tan^{-1} 0 - \tan^{-1}(-1) = 0 - \left(-\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}.$$

Sec.6.8 # 14

8–68 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{x \to 0} \frac{\tan 3x}{\sin 2x}$$

[Solution]

This limit has the form $\frac{0}{0}$. $\displaystyle \lim_{x\to 0} \frac{\tan 3x}{\sin 2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{3\sec^2 3x}{2\cos 2x} = \frac{3(1)^2}{2(1)} = \frac{3}{2}$

Sec.6.8 # 19

8–68 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

[Solution]

This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

Sec.6.8 # 23

8–68 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{t\to 1} \frac{t^6 - 1}{t^3 - 1}$$

[Solution]

This limit has form $\frac{0}{0}$. $\lim_{t\to 1} \frac{t^6 - 1}{t^3 - 1} = \lim_{t\to 1} \frac{6t^5}{3t^2} = \frac{6}{3} = 2$

Sec.6.8 # 26

8–68 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{u\to\infty} \frac{e^{u/10}}{u^3}$$

[Solution]

This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{u\to\infty} \frac{e^{u/10}}{u^3} \overset{\text{H}}{=} \lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{3u^2} \overset{\text{H}}{=} \frac{1}{30}\lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{2u} \overset{\text{H}}{=} \frac{1}{600}\lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{1} = \frac{1}{6000}\lim_{u\to\infty} e^{u/10} = \infty$$