Calculus (II) –Final Exam

1

Prove that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

[Solution]

Let $y = \sin^{-1} x$, then $\sin y = x$ and $-\pi/2 \le y \le \pi/2$.

$\rightarrow \cos y \frac{dy}{dx} = 1$

$\rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$ (now $\cos y \ge 0$, since $-\pi/2 \le y \le \pi/2$)

$\rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$

$\rightarrow \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

2

Differentiate the function.

(a) $h(x) = \ln(x + \sqrt{x^2-1})$. $h'(x) =$ ________.

(b) $f(x) = \ln \ln \ln x$. $f'(x) =$ ________.

(c) $y = \frac{(x^3+1)^4 \sin^2 x}{x^{1/3}}$. $y' =$ ________.

(d) $g(u) = e^{\sqrt{\sec u^2}}$. $g'(u) =$ ________.

(e) $y = (\sin x)^{\ln x}$. $y' =$ ________.

(f) $y = x \log_4(\sin x)$. $y' =$ ________.

(g) $y = x \sin^{-1} x + \sqrt{1-x^2}$. $y' =$ ________.

(h) $y = \tan^{-1}(x - \sqrt{1+x^2})$. $y' =$ ________.

[Solution]

(a)

$$h(x) = \ln\left(x + \sqrt{x^2-1}\right) \quad \Rightarrow \quad h'(x) = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

(b)

$$f(x) = \ln \ln \ln x \quad \Rightarrow \quad f'(x) = \frac{1}{\ln \ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}.$$

$$\text{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty).$$

(c)

$$y = \frac{(x^3+1)^4 \sin^2 x}{x^{1/3}} \quad \Rightarrow \quad \ln|y| = 4\ln\left|x^3+1\right| + 2\ln|\sin x| - \tfrac{1}{3}\ln|x|. \text{ So } \frac{y'}{y} = 4\,\frac{3x^2}{x^3+1} + 2\,\frac{\cos x}{\sin x} - \frac{1}{3x} \quad \Rightarrow$$

$$y' = \frac{(x^3+1)^4 \sin^2 x}{x^{1/3}}\left(\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x}\right).$$

(d)

$$g(u) = e^{\sqrt{\sec u^2}} \quad \Rightarrow$$

$$g'(u) = e^{\sqrt{\sec u^2}} \frac{d}{du} \sqrt{\sec u^2} = e^{\sqrt{\sec u^2}} \frac{1}{2} (\sec u^2)^{-1/2} \frac{d}{du} \sec u^2$$

$$= e^{\sqrt{\sec u^2}} \frac{1}{2\sqrt{\sec u^2}} \cdot \sec u^2 \tan u^2 \cdot 2u = u\sqrt{\sec u^2} \tan u^2 \, e^{\sqrt{\sec u^2}}$$

(e)

$$y = (\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln(\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln x \cdot \ln \sin x \quad \Rightarrow \quad \frac{1}{y} y' = \ln x \cdot \frac{1}{\sin x} \cdot \cos x + \ln \sin x \cdot \frac{1}{x} \quad \Rightarrow$$

$$y' = y\left(\ln x \cdot \frac{\cos x}{\sin x} + \frac{\ln \sin x}{x}\right) \quad \Rightarrow \quad y' = (\sin x)^{\ln x} \left(\ln x \cot x + \frac{\ln \sin x}{x}\right)$$

(f)

$$y = x \log_4 \sin x \quad \Rightarrow \quad y' = x \cdot \frac{1}{\sin x \ln 4} \cdot \cos x + \log_4 \sin x \cdot 1 = \frac{x \cot x}{\ln 4} + \log_4 \sin x$$

(g)

$$y = x \sin^{-1} x + \sqrt{1 - x^2} \quad \Rightarrow$$

$$y' = x \cdot \frac{1}{\sqrt{1 - x^2}} + (\sin^{-1} x)(1) + \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1 - x^2}} + \sin^{-1} x - \frac{x}{\sqrt{1 - x^2}} = \sin^{-1} x$$

(h)

$$y = \tan^{-1}\left(x - \sqrt{x^2 + 1}\right) \quad \Rightarrow$$

$$y' = \frac{1}{1 + \left(x - \sqrt{x^2 + 1}\right)^2} \left(1 - \frac{x}{\sqrt{x^2 + 1}}\right) = \frac{1}{1 + x^2 - 2x\sqrt{x^2 + 1} + x^2 + 1} \left(\frac{\sqrt{x^2 + 1} - x}{\sqrt{x^2 + 1}}\right)$$

$$= \frac{\sqrt{x^2 + 1} - x}{2\left(1 + x^2 - x\sqrt{x^2 + 1}\right)\sqrt{x^2 + 1}} = \frac{\sqrt{x^2 + 1} - x}{2\left[\sqrt{x^2 + 1}\,(1 + x^2) - x(x^2 + 1)\right]} = \frac{\sqrt{x^2 + 1} - x}{2\left[(1 + x^2)\left(\sqrt{x^2 + 1} - x\right)\right]}$$

$$= \frac{1}{2(1 + x^2)}$$

 # 3

(a) $e^{2x} - 3e^x + 2 = 0$, $x =$ ________.

(b) $\ln(2x + 1) = 2 - \ln x$, $x =$ ________.

[Solution]

(a)

$e^{2x} - 3e^x + 2 = 0 \;\Leftrightarrow\; (e^x - 1)(e^x - 2) = 0 \;\Leftrightarrow\; e^x = 1$ or $e^x = 2 \;\Leftrightarrow\; x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

(b)

$$\ln(2x + 1) = 2 - \ln x \quad \Rightarrow \quad \ln x + \ln(2x + 1) = \ln e^2 \quad \Rightarrow \quad \ln[x(2x + 1)] = \ln e^2 \quad \Rightarrow \quad 2x^2 + x = e^2 \quad \Rightarrow$$

$$2x^2 + x - e^2 = 0 \quad \Rightarrow \quad x = \frac{-1 + \sqrt{1 + 8e^2}}{4} \quad [\text{since } x > 0].$$

 # 4

Find the inverse function of $y = \frac{1 - e^{-x}}{1 + e^{-x}}$.

[Solution]

$y = f(x) = \dfrac{1-e^{-x}}{1+e^{-x}} \Rightarrow y(1+e^{-x}) = 1-e^{-x} \Rightarrow y + ye^{-x} = 1 - e^{-x} \Rightarrow ye^{x} + y = e^{x} - 1$ [multiply each term by e^{x}] $\Rightarrow ye^{x} - e^{x} = -y-1 \Rightarrow e^{x}(y-1) = -y-1 \Rightarrow e^{x} = \dfrac{1+y}{1-y} \Rightarrow x = \ln\left(\dfrac{1+y}{1-y}\right)$.

Interchange x and y: $y = \ln\left(\dfrac{1+x}{1-x}\right)$. So $f^{-1}(x) = \ln\left(\dfrac{1+x}{1-x}\right)$.

5

Find the limit.

(a) $\lim\limits_{x\to\infty}[\ln(2+x) - \ln(1+x)] =$__________.

(b) $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{\cos x - (x+1)} =$__________.

(c) $\lim\limits_{x\to\infty} \cos^{-1}\left(\dfrac{1+x^2}{1+2x^2}\right) =$__________.

(d) $\lim\limits_{x\to 0} \dfrac{x}{\tan^{-1}(4x)} =$__________.

(e) $\lim\limits_{x\to 0^+}(\sin x)(\ln x) =$__________.

(f) $\lim\limits_{x\to 1}(2-x)^{\tan(\pi x/2)} =$__________.

[Solution]

(a)

$$\lim_{x\to\infty}[\ln(2+x) - \ln(1+x)] = \lim_{x\to\infty}\ln\left(\frac{2+x}{1+x}\right) = \lim_{x\to\infty}\ln\left(\frac{2/x+1}{1/x+1}\right) = \ln\frac{1}{1} = \ln 1 = 0$$

(b)

This limit has form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\cos x - (x+1)} = \lim\limits_{x\to 0}\dfrac{2e^{2x}}{-\sin x - 1} = \dfrac{2(1)}{-1} = -2$

(c)

Let $t = \dfrac{1+x^2}{1+2x^2}$. As $x\to\infty$, $t = \dfrac{1+x^2}{1+2x^2} = \dfrac{1/x^2+1}{1/x^2+2} \to \dfrac{1}{2}$

$$\lim_{x\to\infty}\arccos\left(\frac{1+x^2}{1+2x^2}\right) = \lim_{t\to 1/2}\arccos t = \arccos\tfrac{1}{2} = \tfrac{\pi}{3}.$$

(d)

This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{x}{\tan^{-1}(4x)} \overset{H}{=} \lim\limits_{x\to 0}\dfrac{1}{\dfrac{1}{1+(4x)^2}\cdot 4} = \lim\limits_{x\to 0}\dfrac{1+16x^2}{4} = \dfrac{1}{4}$

(e)

This limit has the form $0\cdot(-\infty)$.

$$\lim_{x\to 0^+}\sin x\,\ln x = \lim_{x\to 0^+}\frac{\ln x}{\csc x} \overset{H}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc x\,\cot x} = -\lim_{x\to 0^+}\left(\frac{\sin x}{x}\cdot\tan x\right) = -\left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^+}\tan x\right)$$
$$= -1\cdot 0 = 0$$

(f)

$y = (2-x)^{\tan(\pi x/2)} \quad \Rightarrow \quad \ln y = \tan\left(\frac{\pi x}{2}\right)\ln(2-x) \quad \Rightarrow$

$$\lim_{x\to1}\ln y = \lim_{x\to1}\left[\tan\left(\frac{\pi x}{2}\right)\ln(2-x)\right] = \lim_{x\to1}\frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)} \overset{\text{H}}{=} \lim_{x\to1}\frac{\frac{1}{2-x}(-1)}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} = \frac{2}{\pi}\lim_{x\to1}\frac{\sin^2\left(\frac{\pi x}{2}\right)}{2-x}$$

$$= \frac{2}{\pi}\cdot\frac{1^2}{1} = \frac{2}{\pi} \quad \Rightarrow \quad \lim_{x\to1}(2-x)^{\tan(\pi x/2)} = \lim_{x\to1}e^{\ln y} = e^{(2/\pi)}$$

6

Evaluate the integral.

(a) $\int \frac{\cos(\ln t)}{t}\,dt$ =________.

(b) $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$ =________.

(c) $\int \frac{\log_{10}x}{x}\,dx$ =________.

(d) $\int_0^{\sqrt{3}/4} \frac{1}{1+16x^2}\,dx$ =________.

(e) $\int t\sec^2(2t)\,dt$ =________.

(f) $\int x\tan^2 x\,dx$ =________.

(g) $\int_0^{\pi}\cos^4(2t)\,dt$ =________.

(h) $\int \tan^2\theta\sec^4\theta\,d\theta$ =________.

(i) $\int_0^{1/2} x\sqrt{1-4x^2}\,dx$ =________.

(j) $\int_0^1 \sqrt{x^2+1}\,dx$ =________.

[Solution]

(a)

Let $u = \ln t$. Then $du = \frac{1}{t}\,dt$, so $\int \frac{\cos(\ln t)}{t}\,dt = \int \cos u\,du = \sin u + C = \sin(\ln t) + C$.

(b)

Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

(c)

$\int \frac{\log_{10}x}{x}\,dx = \int \frac{(\ln x)/(\ln 10)}{x}\,dx = \frac{1}{\ln 10}\int \frac{\ln x}{x}\,dx$. Now put $u = \ln x$, so $du = \frac{1}{x}\,dx$, and the expression becomes

$\frac{1}{\ln 10}\int u\,du = \frac{1}{\ln 10}\left(\frac{1}{2}u^2 + C_1\right) = \frac{1}{2\ln 10}(\ln x)^2 + C.$

Or: The substitution $u = \log_{10}x$ gives $du = \frac{dx}{x\ln 10}$ and we get $\int \frac{\log_{10}x}{x}\,dx = \frac{1}{2}\ln 10(\log_{10}x)^2 + C$.

(d)

Let $u = 4x$. Then $du = 4\,dx$, so

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{4}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \tfrac{1}{4}\left[\tan^{-1} u\right]_0^{\sqrt{3}} = \tfrac{1}{4}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 0\right) = \tfrac{1}{4}\left(\tfrac{\pi}{3} - 0\right) = \tfrac{\pi}{12}.$$

(e)

Let $u = t$, $dv = \sec^2 2t\,dt \quad \Rightarrow \quad du = dt$, $v = \frac{1}{2}\tan 2t$. Then

$$\int t\sec^2 2t\,dt = \tfrac{1}{2}t\tan 2t - \tfrac{1}{2}\int \tan 2t\,dt = \tfrac{1}{2}t\,\tan 2t - \tfrac{1}{4}\ln|\sec 2t| + C.$$

(f)

$\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$. Let $u = x$, $dv = \sec^2 x\,dx \quad \Rightarrow \quad du = dx$, $v = \tan x$.

Then by Equation 2, $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x|$, and thus,

$\int x\tan^2 x\,dx = x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C.$

(g)

$$\begin{aligned}\int_0^{\pi} \cos^4(2t)\,dt &= \int_0^{\pi}[\cos^2(2t)]^2\,dt = \int_0^{\pi}\left[\tfrac{1}{2}(1+\cos(2\cdot 2t))\right]^2 dt \qquad \text{[half-angle identity]}\\ &= \tfrac{1}{4}\int_0^{\pi}[1+2\cos 4t+\cos^2(4t)]\,dt = \tfrac{1}{4}\int_0^{\pi}[1+2\cos 4t+\tfrac{1}{2}(1+\cos 8t)]\,dt\\ &= \tfrac{1}{4}\int_0^{\pi}\left(\tfrac{3}{2}+2\cos 4t+\tfrac{1}{2}\cos 8t\right)\,dt = \tfrac{1}{4}\left[\tfrac{3}{2}t+\tfrac{1}{2}\sin 4t+\tfrac{1}{16}\sin 8t\right]_0^{\pi} = \tfrac{1}{4}\left[\left(\tfrac{3}{2}\pi+0+0\right)-0\right] = \tfrac{3}{8}\pi\end{aligned}$$

(h)

$$\begin{aligned}\int \tan^2\theta\,\sec^4\theta\,d\theta &= \int \tan^2\theta\,\sec^2\theta\,\sec^2\theta\,d\theta = \int \tan^2\theta\,(\tan^2\theta+1)\,\sec^2\theta\,d\theta\\ &= \int u^2(u^2+1)\,du \qquad [u=\tan\theta,\ du=\sec^2\theta\,d\theta]\\ &= \int (u^4+u^2)\,du = \tfrac{1}{5}u^5+\tfrac{1}{3}u^3+C = \tfrac{1}{5}\tan^5\theta+\tfrac{1}{3}\tan^3\theta+C\end{aligned}$$

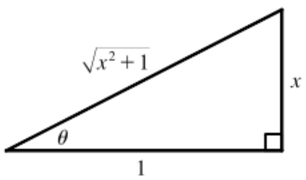
(i)

$$\begin{aligned}\int_0^{1/2} x\sqrt{1-4x^2}\,dx &= \int_1^0 u^{1/2}\left(-\tfrac{1}{8}\,du\right) \qquad \begin{bmatrix} u=1-4x^2, \\ du=-8x\,dx \end{bmatrix}\\ &= \tfrac{1}{8}\left[\tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{1}{12}(1-0) = \tfrac{1}{12}\end{aligned}$$

(j)

. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$,

$\sqrt{x^2+1} = \sec\theta$ and $x = 0 \quad \Rightarrow \quad \theta = 0$, $x = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$, so



$$\begin{aligned}\int_0^1 \sqrt{x^2+1}\,dx &= \int_0^{\pi/4}\sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta\\ &= \tfrac{1}{2}\Big[\sec\theta\,\tan\theta+\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4} \qquad \text{[by Example 7.2.8]}\\ &= \tfrac{1}{2}\left[\sqrt{2}\cdot 1+\ln(1+\sqrt{2})-0-\ln(1+0)\right] = \tfrac{1}{2}\left[\sqrt{2}+\ln(1+\sqrt{2})\right]\end{aligned}$$