Calculus (II) – Midterm Exam

 # 1

Suppose f is continuous on [a, b].

(a) If $g(x)$ = ________, then $g'(x) = f(x)$.

(b) $\int_a^b f(x)dx =$________, where F is any antiderivative of f.

[Solution]

(a) $\int_a^x f(t)dt$

(b)F(b)−F(a)

 # 2

Determine a region whose area is equal to $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$. (Do not evaluate the limit.)

[Solution]

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$ can be interpreted as the area of the region lying under the graph of $y=\sqrt{1+x}$ on the interval $[0,3]$, since for $y=\sqrt{1+x}$ on $[0,3]$ with $\Delta x = \frac{3-0}{n} = \frac{3}{n}$, $x_i = 0 + i\,\Delta x = \frac{3i}{n}$, and $x_i^* = x_i$, the expression for the area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\frac{3i}{n}}\,\frac{3}{n}$. Note that this answer is not unique. We could use $y=\sqrt{x}$ on $[1,4]$ or, in general, $y=\sqrt{x-n}$ on $[n+1, n+4]$, where n is any real number.

 # 3

Express $\int_2^5 (x^2+\frac{1}{x})dx$ as a limit of Riemann sums. (Do not evaluate the limit.)

[Solution]

$f(x) = x^2 + \frac{1}{x}$, $a=2$, $b=5$, and $\Delta x = \frac{5-2}{n} = \frac{3}{n}$. Using Theorem 4, we get $x_i^* = x_i = 2 + i\,\Delta x = 2 + \frac{3i}{n}$, so

$$\int_2^5 \left(x^2+\frac{1}{x}\right)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(2+\frac{3i}{n}\right)^2 + \frac{1}{2+\frac{3i}{n}}\right]\cdot\frac{3}{n}.$$

 # 4

Evaluate the derivative.

(a) $R(y) = \int_y^2 t^3 \sin t\, dt$. $R'(y)$ = ________.

(b) $h(x) = \int_{\sqrt{x}}^{x^3} \cos(t^2)\, dt$. $h'(x)$ = ________.

(c) $y = \int_1^{3x+2} \frac{t}{1+t^3} dt$. y' = ________.

[Solution]

(a)

$$R(y)=\int_y^2 t^3\sin t\,dt=-\int_2^y t^3\sin t\,dt \quad\Rightarrow\quad R'(y)=-\frac{d}{dy}\int_2^y t^3\sin t\,dt=-y^3\sin y$$

(b)

$$h(x)=\int_{\sqrt{x}}^{x^3}\cos(t^2)\,dt=\int_{\sqrt{x}}^{0}\cos(t^2)\,dt+\int_0^{x^3}\cos(t^2)\,dt=-\int_0^{\sqrt{x}}\cos(t^2)\,dt+\int_0^{x^3}\cos(t^2)\,dt \quad\Rightarrow$$

$$h'(x)=-\cos\left((\sqrt{x})^2\right)\cdot\frac{d}{dx}(\sqrt{x})+[\cos(x^3)^2]\cdot\frac{d}{dx}(x^3)=-\frac{1}{2\sqrt{x}}\cos x+3x^2\cos(x^6)$$

(c)

Let $u=3x+2$. Then $\frac{du}{dx}=3$. Also, $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, so

$$y'=\frac{d}{dx}\int_1^{3x+2}\frac{t}{1+t^3}\,dt=\frac{d}{du}\int_1^u\frac{t}{1+t^3}\,dt\cdot\frac{du}{dx}=\frac{u}{1+u^3}\frac{du}{dx}=\frac{3x+2}{1+(3x+2)^3}\cdot 3=\frac{3(3x+2)}{1+(3x+2)^3}$$

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Evaluate the integral.

(a) $\int_{-1}^{2}(3u-2)(u+1)du=$ ________.

(b) $\int_0^{\pi/6}\frac{\sin t}{\cos^2 t}dt=$ ________.

(c) $\int_{-\pi/3}^{\pi/3}x^4\sin x dx=$ ________.

(d) $\int\frac{x^3-2\sqrt{x}}{x}dx=$ ________.

(e) $\int\frac{\sin 2x}{\sin x}dx=$ ________.

(f) $\int_0^2|2x-1|dx=$ ________.

(g) $\int\frac{1}{\cos^2 t\sqrt{1+\tan t}}dt=$ ________.

(h) $\int \sin t\sec^2(\cos t)\,dt=$ ________.

[Solution]

(a)

$$\int_{-1}^{2}(3u-2)(u+1)\,du=\int_{-1}^{2}(3u^2+u-2)\,du=\left[u^3+\tfrac{1}{2}u^2-2u\right]_{-1}^{2}=(8+2-4)-\left(-1+\tfrac{1}{2}+2\right)=6-\tfrac{3}{2}=\tfrac{9}{2}$$

(b)

Let $u=\cos t$, so $du=-\sin t\,dt$. When $t=0$, $u=1$; when $t=\frac{\pi}{6}$, $u=\sqrt{3}/2$. Thus,

$$\int_0^{\pi/6}\frac{\sin t}{\cos^2 t}\,dt=\int_1^{\sqrt{3}/2}\frac{1}{u^2}(-du)=\left[\frac{1}{u}\right]_1^{\sqrt{3}/2}=\frac{2}{\sqrt{3}}-1.$$

(c)

$\int_{-\pi/3}^{\pi/3}x^4\sin x\,dx=0$ by Theorem 6(b), since $f(x)=x^4\sin x$ is an odd function.

(d)

$$\int \frac{x^3 - 2\sqrt{x}}{x}\,dx = \int \left(\frac{x^3}{x} - \frac{2x^{1/2}}{x}\right)dx = \int (x^2 - 2x^{-1/2})\,dx = \frac{x^3}{3} - 2\,\frac{x^{1/2}}{1/2} + C = \tfrac{1}{3}x^3 - 4\sqrt{x} + C$$

(e)

$$\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\,\cos x}{\sin x}\,dx = \int 2\cos x\,dx = 2\sin x + C$$

(f)

$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \ge 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \ge \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}$$

Thus, $\int_0^2 |2x-1|\,dx = \int_0^{1/2}(1-2x)\,dx + \int_{1/2}^2 (2x-1)\,dx = \left[x - x^2\right]_0^{1/2} + \left[x^2 - x\right]_{1/}^{2}$

$$= \left(\tfrac{1}{2} - \tfrac{1}{4}\right) - 0 + (4-2) - \left(\tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{1}{4} + 2 - \left(-\tfrac{1}{4}\right) = \tfrac{5}{2}$$

(g)

Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$, so

$$\int \frac{dt}{\cos^2 t\,\sqrt{1+\tan t}} = \int \frac{\sec^2 t\,dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

(h)

Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$\int \sin t\,\sec^2(\cos t)\,dt = \int \sec^2 u \cdot (-du) = -\tan u + C = -\tan(\cos t) + C.$

 # 6

Sketch the region enclosed by $y = x - 1$ and $y^2 = 2x + 6$, and find its area.

[Solution]

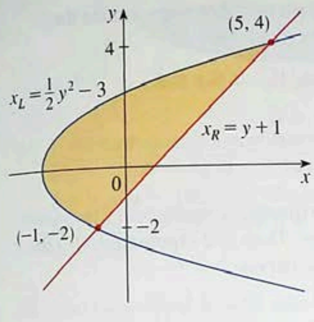

FIGURE 13

EXAMPLE 6 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION By solving the two equations we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for x and notice from Figure 13 that the left and right boundary curves are

$$x_L = \tfrac{1}{2}y^2 - 3 \qquad x_R = y + 1$$

We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$\begin{aligned} A &= \int_{-2}^{4} (x_R - x_L)\,dy \\ &= \int_{-2}^{4} \left[(y+1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy \\ &= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy \\ &= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\Bigg]_{-2}^{4} \\ &= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8\right) = 18 \end{aligned}$$

□

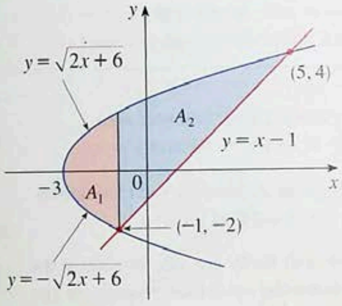

FIGURE 14

We could have found the area in Example 6 by integrating with respect to x instead of y, but the calculation is much more involved. It would have meant splitting the region in two and computing the areas labeled A_1 and A_2 in Figure 14. The method we used in Example 6 is *much* easier.

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Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 1$, $x = 2$ about $y = -3$.

[Solution]

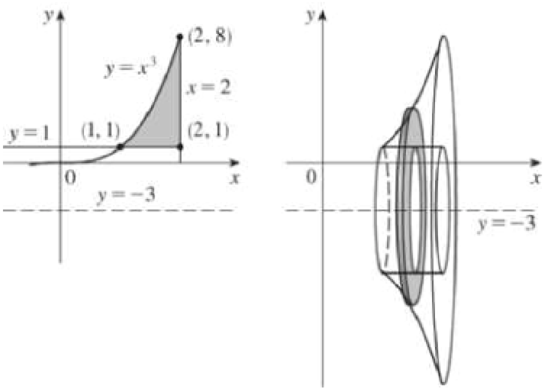
A cross-section is a washer with inner radius $1 - (-3) = 4$ and outer radius $x^3 - (-3) = x^3 + 3$, so its area is

$$A(x) = \pi(x^3+3)^2 - \pi(4)^2 = \pi(x^6 + 6x^3 - 7).$$

$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi(x^6 + 6x^3 - 7)\,dx$$

$$= \pi\left[\tfrac{1}{7}x^7 + \tfrac{3}{2}x^4 - 7x\right]_1^2$$

$$= \pi\left[\left(\tfrac{128}{7} + 24 - 14\right) - \left(\tfrac{1}{7} + \tfrac{3}{2} - 7\right)\right] = \tfrac{471\pi}{14}$$

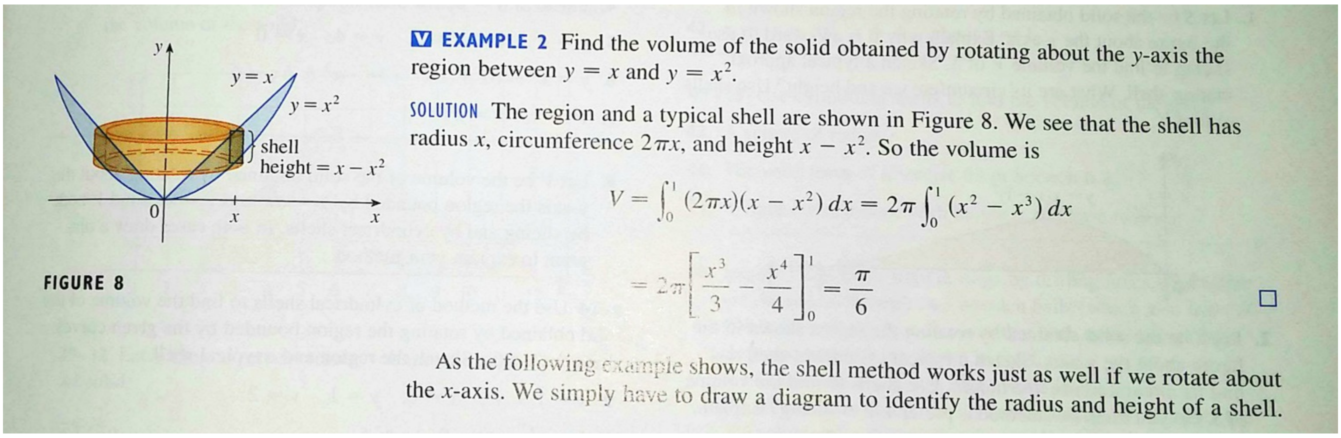


8

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

[Solution]

<解 1>



EXAMPLE 2 Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

SOLUTION The region and a typical shell are shown in Figure 8. We see that the shell has radius x, circumference $2\pi x$, and height $x - x^2$. So the volume is

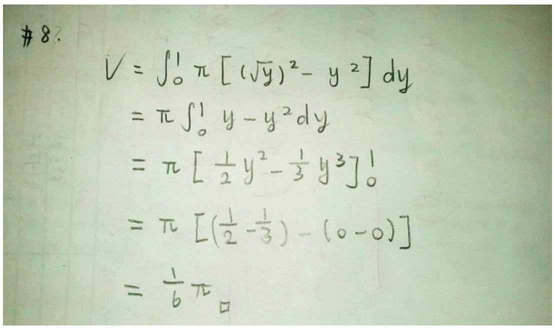
$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$ □

As the following example shows, the shell method works just as well if we rotate about the x-axis. We simply have to draw a diagram to identify the radius and height of a shell.

FIGURE 8

<解 2>



#8.

$$V = \int_0^1 \pi\left[(\sqrt{y})^2 - y^2\right] dy$$

$$= \pi \int_0^1 y - y^2\, dy$$

$$= \pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1$$

$$= \pi\left[\left(\tfrac{1}{2} - \tfrac{1}{3}\right) - (0 - 0)\right]$$

$$= \tfrac{1}{6}\pi$$ □

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$f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}$. $f^{-1}(x)$= ________.

[Solution]

For $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$, the domain is $x \geq 0$. $f(0) = 1$ and as x increases, y decreases. As $x \to \infty$,

$\dfrac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \dfrac{1/\sqrt{x}-1}{1/\sqrt{x}+1} \to \dfrac{-1}{1} = -1$, so the range of f is $-1 < y \leq 1$. Thus, the domain of f^{-1} is $-1 < x \leq 1$.

$y = \dfrac{1-\sqrt{x}}{1+\sqrt{x}} \Rightarrow y(1+\sqrt{x}) = 1-\sqrt{x} \Rightarrow y + y\sqrt{x} = 1-\sqrt{x} \Rightarrow \sqrt{x} + y\sqrt{x} = 1-y \Rightarrow$

$\sqrt{x}(1+y) = 1-y \Rightarrow \sqrt{x} = \dfrac{1-y}{1+y} \Rightarrow x = \left(\dfrac{1-y}{1+y}\right)^2$. Interchange x and y: $y = \left(\dfrac{1-x}{1+x}\right)^2$. So

$f^{-1}(x) = \left(\dfrac{1-x}{1+x}\right)^2$ with $-1 < x \leq 1$.

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$f(x) = 3 + x^2 + \tan(\frac{\pi x}{2})$ for $-1 < x < 1$.

$(f^{-1})'(3)$ =________.

[Solution]

$f(0) = 3 \Rightarrow f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \Rightarrow f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and

$f'(0) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$. Thus, $(f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(0) = 2/\pi$.

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A particle moves along a line as that its velocity at time t is $v(t) = t^2 - t - 6$ (m/s). Find the distance traveled during the time period $1 \leq t \leq 4$.

[Solution]

(b) Note that $v(t) = t^2 - t - 6 = (t-3)(t+2)$ and so $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$ on $[3, 4]$. Thus, from Equation 3, the distance traveled is

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6} \approx 10.17 \text{ m}$$

□

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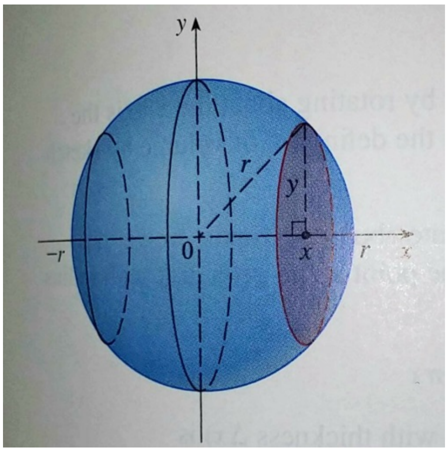
Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

[Solution]



The cross-sectional area is $A(x)$

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Using the definition of volume with $a = -r$ and $b = r$, we have

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= 2\pi \int_{0}^{r} (r^2 - x^2)\,dx \qquad \text{(The integrand is even.)}$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right)$$